Nondecay probability of the ''correct'' state of a memory cell: Analytic approach versus numeric simulation

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This paper presents a complete description of noise-induced transitions in a memory cell based on the parametric quantron (a superconducting ring enclosed by a Josephson junction). The time dependence of the nondecay probability of the ''correct'' state of a memory cell is found to follow exponential behavior even for a large noise intensity compared to a potential barrier height. [S1063-651X(98)08310-X]

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I. INTRODUCTION

It is well known that the use of HTS overdamped Josephson junctions as logic devices and memory cells is quite prospective because of low cooling costs and high operating frequencies (see, e.g., [1,2], and also proceedings of the last ASC, ISEC, and EUCAS conferences). However, higher operation temperatures lead to higher noise levels and an increase in thermally induced switching errors. One of the problems arising in the design of HTS Josephson devices is the absence of a complete theory of noise-induced transitions in nonlinear systems, valid for arbitrary noise intensity. Moreover, the difference between the theoretically predicted (on the basis of approximate approaches [3]) and the experimentally observed switching probabilities of a Josephson balanced comparator has been recently demonstrated for temperatures above 25 K [2].

The parametric quantron (PQ) [4], consisting of a superconducting ring enclosed by a Josephson junction (JJ), is used for storage of information as the main element of memory cells. When the PQ is applied as a bistable memory cell, one of the most important parameters is immunity to thermal fluctuations, which can spontaneously switch the memory cell from one state to another. However, because of mathematical difficulties, only approximate evaluations have been done for such important characteristics as the mean decay time of the ''correct'' state (or mean time of spontaneous switching), while the variance and higher cumulants of the decay time have not been investigated. Also, it is known that, if the ratio between a potential barrier separating stable states and noise intensity is high, the probability that decay of the ''correct'' state of the memory cell will not occur until some time t (nondecay probability) is proportional to the exponent: $\sim\exp(-t/\vartheta_K)$ [5], where $\vartheta_K=\tau_K/2$ and τ_K is the approximate mean decay time of the ''correct'' state (Kramers' time) [4,6].

The present paper is devoted to the investigation of immunity to thermal fluctuations of a bistable memory cell based on the parametric quantron. The aim of the present paper is to obtain exact time characteristics (moments of decay time) valid for an arbitrary noise intensity, and to find the boundaries of validity of the exponential approximation of the nondecay probability. However, having compared exponential approximation with the computer simulation results, we have found a really close coincidence even for a large noise intensity, if ϑ_K in the factor of the exponent is replaced by the exact mean decay time τ. The analysis has been done using the well-known resistive model of a Josephson junction (JJ) [7], where quantum effects are not included.

II. MAIN EQUATIONS AND SETUP OF THE PROBLEM

It is known that the dynamics of an overdamped single junction superconducting quantum interference device (SQUID) (''the parametric quantron'' [4]) with fluctuations taken into account is well described by the Langevin equation:

$$\omega_c^{-1}\frac{d\varphi(t)}{dt}=-\frac{du(\varphi)}{d\varphi}-i_F(t), \tag{1}$$

where

$$u(\varphi)=1-\cos\varphi+(\varphi-\varphi_e)^2/2\ell \tag{2}$$

is the dimensionless potential profile, $\varphi=2\pi\Phi/\Phi_0$ is the dimensionless flux through the ring, Φ_0 is the flux quantum, the quantity φ_e describes the external flux, $\ell=L/L_0$, L is the inductance of the ring, $L_0=\Phi_0/2\pi I_c$, I_c is the critical current of the junction, $i_F(t)=I_F/I_c$, I_F is the random component of the current, $\omega_c=2\pi R_N I_c/\Phi_0$ is the characteristic frequency of the JJ, and $R_N^{-1}=G_N$ is the normal conductivity of the JJ. In the case where only thermal fluctuations are taken into account, the random current may be represented by the white Gaussian noise:

$$\langle i_F(t)\rangle=0,\quad \langle i_F(t)i_F(t+\tau)\rangle=\frac{2\gamma}{\omega_c}\delta(\tau),$$

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where $\gamma = 2\pi kT/\Phi_0 I_c = I_T/I_c$ is the dimensionless noise intensity, T is the temperature, and k is the Boltzmann constant.

In the case of nonzero noise intensity, the flux φ is a random quantity described by the transitional probability density $W(\varphi,t)$. It is well known that the Fokker-Planck equation (FPE) for the probability density $W(\varphi,t)$ corresponds to Eq. (1) for the flux:

$$\frac{\partial W(\varphi,t)}{\partial t} = -\frac{\partial G(\varphi,t)}{\partial \varphi} = \omega_c \frac{\partial}{\partial \varphi}\left\{\left[\frac{du(\varphi)}{d\varphi} W(\varphi,t)\right] + \gamma \frac{\partial W(\varphi,t)}{\partial \varphi}\right\}, \quad (3)$$

where $G(\varphi,t)$ is the probability current. The initial and boundary conditions for Eq. (3) with the potential (2) are

$$W(\varphi,0) = \delta(\varphi - \varphi_0) \quad \text{and} \quad G(\pm\infty,t) = 0. \quad (4)$$

The nonlinear dynamical system described by the potential profile (2) may have one or several stable states depending on quantities of parameters ℓ and φ_e. To use the parametric quantron as a memory cell, it is enough to have two stable states [4], which may be realized at $3 < \ell < 8$, $\varphi_e = \pi$. In this particular case the potential (2) has the parabolic barrier separating two wells with parabolic walls, slightly modulated by the cosine term. However, it is difficult to analyze the functioning of such a system because the solution of FPE (3) for the potential (2) is unknown. When the potential barrier separating the stable states is high enough in comparison with the noise intensity, $\Delta u \gg \gamma$, it is possible to approximately obtain the mean decay time of the ''correct'' state [4] on the basis of the Kramers method [6] ($\alpha = \Delta u/\gamma$):

$$\tau_K = \frac{2\pi}{b\omega_c} e^{\alpha}, \quad \alpha \gg 1 \quad (5)$$

where $b = \sqrt{b_{\min} b_{\max}}$, and $b_{\min}$ and $b_{\max}$ are the curvatures of the bottom of the well and the top of the barrier of the potential (2), respectively. In this case the probability $P(\varphi_0,t)$ that decay of the ''correct'' state will not occur until some time t may be very well approximated by the exponent:

$$P(\varphi_0,t) = \frac{\exp(-t/\vartheta_K) + 1}{2}, \quad \vartheta_K = \tau_K/2. \quad (6)$$

Here φ_0 is the coordinate of the initial δ-shaped probability distribution. Certainly, in the case (5), (6) the probability does not depend on φ_0, but searching further for the exact probability evolution we will keep in mind the initial distribution at φ_0. Formula (6) may be obtained via the approach presented in the book by Gardiner [5].

Let, initially, a bit of information be stored in the memory cell by locating the phase point in the left minimum of the potential (2), such that $\varphi_0 \leqslant \pi$. The decay time of the ''correct'' state we define as the time needed to cross the barrier top $\varphi = \pi$. It is clear that this decay time is a random value and the problem is how to obtain its mean $\tau = \kappa_1 = \langle t \rangle$, variance $D = \kappa_2 = \langle t^2 \rangle - \langle t \rangle^2$, and higher cumulants κ_n.

The required time characteristics may be introduced from the probability $P(\varphi_0,t)$ that transition of the phase point from φ_0 outside the considered interval $(-\infty,\pi)$ will not occur during the time $t > 0$: $P(\varphi_0,t) = \int_{-\infty}^{\pi} W(\varphi,t)\,d\varphi$.

By analogy to moments of the first passage time (FPT) [8] we can introduce the moments $\vartheta_n(\varphi_0,\pi) = \vartheta_n$ of transition time [9], bearing in mind that even for an infinitely long time the phase point may still be located within the considered interval because $\lim_{t\to\infty} P(\varphi_0,t) = 1/2$:

$$\vartheta_n = \langle t^n \rangle = \frac{\int_0^{\infty} t^n \{\partial P(\varphi_0,t)/\partial t\}\,dt}{P(\varphi_0,\infty) - P(\varphi_0,0)}, \quad (7)$$

where $P(\varphi_0,\infty) - P(\varphi_0,0) = \int_0^{\infty} \{\partial P(\varphi_0,t)/\partial t\}\,dt$ is the factor of normalization. In our particular case $P(\varphi_0,0) = 1$, $P(\varphi_0,\infty) = 1/2$. Here we can formally denote the derivative of the probability divided by the factor of normalization as $w(\varphi_0,t)$ and thus introduce the probability density of transition time in the following way:

$$w(\varphi_0,t) = \frac{\partial P(\varphi_0,t)}{\partial t[P(\varphi_0,\infty) - P(\varphi_0,0)]}. \quad (8)$$

It is easy to check that the normalization condition is satisfied given such a definition, $\int_0^{\infty} w(\varphi_0,t)\,dt = 1$. The condition of nonnegativity of the probability density $w(\varphi_0,t) \geqslant 0$ is, actually, the monotonic condition of the probability $P(\varphi_0,t)$.

The above-mentioned cumulants of transition time κ_n are much more useful for our purpose to construct the probability $P(\varphi_0,t)$ that is the integral transformation of the just introduced probability density of transition time $w(\varphi_0,t)$. Unlike the representation via moments, the Fourier transformation of the probability density — the characteristic function — decomposed into a set of cumulants may be inversely transformed into the required probability density. The representation of κ_n via moments ϑ_n is described in the book by Malakhov [10] (see also [11]).

III. EXACT TIME CHARACTERISTICS

The method used is based on the Laplace transformation of the FPE (3) and obtaining time characteristics of $W(\varphi,t)$ evolution immediately from the solution of the equation for probability density Laplace transform [9,12] $Y(\varphi,s) = Y(\varphi) = \int_0^{\infty} W(\varphi,t) e^{-st}\,dt$:

$$\frac{d^2 Y(\varphi)}{d\varphi^2} + \frac{d}{d\varphi}\left[\frac{du(\varphi)}{\gamma d\varphi} Y(\varphi)\right] - sBY(\varphi) = -B\delta(\varphi - \varphi_0), \quad (9)$$

where $B = 1/\gamma\omega_c$.

It is known that there is the recurrent formula [13] for moments of the FPT of the boundary located at $\varphi = c > \varphi_0$ by the phase point under noise perturbation $[u(-\infty) = +\infty]$:

$$T_n(\varphi_0,c) = nB \int_{\varphi_0}^{c} e^{u(\varphi)/\gamma} \int_{-\infty}^{\varphi} T_{n-1}(x,c) e^{-u(x)/\gamma}\,dx\,d\varphi, \quad (10)$$

which represents the nth moment of the FPT directly from the function of the potential profile $u(\varphi)$ and the $(n-1)$th moment. Here $T_0(\varphi_0,c)=1$ and $T_1(\varphi_0,c)$ is the mean FPT.

Using the duality of time characteristics, proved in [9], it can be demonstrated that all moments of transition time ϑ_n in a symmetric potential over a point of symmetry coincide with the corresponding moments of the first passage time of the boundary located at the point of symmetry: $\vartheta_n(\varphi_0,c)=T_n(\varphi_0,c)$ (in our particular case $c=\pi$). Thus formula (10) is also valid for moments of transition time. When the intensity of thermal fluctuations is much smaller than the barrier height, $\gamma \ll \Delta u$, the following asymptotic representation can be obtained from formula (10) for moments of transition time:

$$\vartheta_n(\varphi_0,c)=n!\,\vartheta_1^n(\varphi_0,c), \quad \Delta u \gg \gamma. \tag{11}$$

The results of computer simulation demonstrate that expression (11) is valid up to $\Delta u/\gamma \geqslant 2$. Using the properties of cumulants [10], similar representation can be obtained for κ_n:

$$\kappa_n(\varphi_0,c)=(n-1)!\,\kappa_1^n(\varphi_0,c), \quad \Delta u \gg \gamma. \tag{12}$$

It is known that the characteristic function $\Theta(\varphi_0,\omega)=\int_0^\infty w(\varphi_0,t)e^{j\omega t}dt$ $(j=\sqrt{-1})$ can be represented as a set of cumulants:

$$\Theta(\varphi_0,\omega)=\exp\left[\sum_{n=1}^{\infty}\frac{\kappa_n(\varphi_0,c)}{n!}(j\omega)^n\right].$$

For our particular case (12) this set can be summarized and inverse Fourier transformed, so we get

$$w(\varphi_0,t)=\frac{e^{-t/\tau}}{\tau}, \quad \Delta u \gg \gamma \tag{13}$$

where τ is the mean transition time $[\tau(\varphi_0,\pi)\equiv\vartheta_1\equiv\kappa_1]$:

$$\tau(\varphi_0,\pi)=\frac{1}{\gamma\omega_c}\int_{\varphi_0}^{\pi}e^{u(\varphi)/\gamma}\int_{-\infty}^{\varphi}e^{-u(x)/\gamma}dx\,d\varphi, \tag{14}$$

with the asymptotic representation $(\alpha=\Delta u/\gamma)$:

$$\tau(\varphi_0,\pi)=\tau=\frac{\pi}{b\,\omega_c}e^{\alpha}, \quad \alpha \gg 1. \tag{15}$$

Formula (13) is, unfortunately, not valid for small periods of time $t \ll \tau$, because it assumes that a quasi-steady-state distribution in the initial well is already reached and then the escape over the barrier happens, so the initial transition to the quasi-steady-state is neglected. Namely, this circumstance will lead to a slight distinction of the numerically simulated probability from its exponential approximation (see the next section). The asymptotic representation of the probability density $w(\varphi_0,t)$ for small periods of time was obtained in [14] and it has been demonstrated that the time of transition to a quasi-steady-state in the initial well is really much smaller than the mean decay time (15).

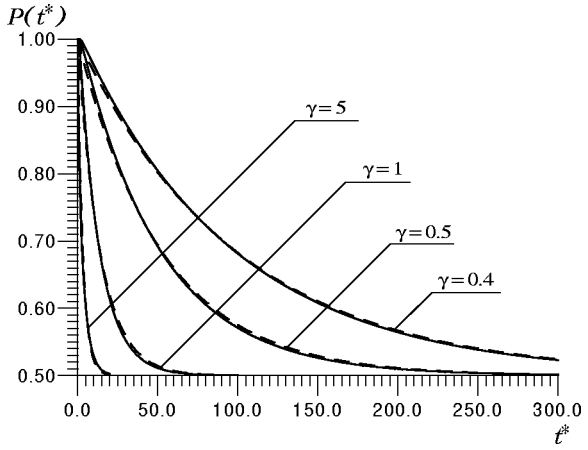

FIG. 1. Evolution of the nondecay probability for different values of noise intensity.

IV. PROBABILITY EVOLUTION

Integrating probability density (13), with the account of definition (8), we get the following expression for the probability $P(\varphi_0,t)$ that decay of the ''correct'' state of a memory cell will not occur until some time t $(\alpha=\Delta u/\gamma)$:

$$P(\varphi_0,t)=\frac{\exp(-t/\tau)+1}{2}, \quad \alpha \gg 1 \tag{16}$$

where the mean decay time τ (15) is two times smaller than the Kramers' time (3) and thus formula (16) completely coincides with (6). Actually, the validity of formula (16) coincides with the validity of formula (15) and, as previous calculations demonstrate, formula (15) is valid up to $\alpha \geqslant 2$. Our aim was to numerically test this fact for formula (16). We used the usual explicit difference scheme to solve the Fokker-Planck equation (3), assuming the reflecting boundary conditions $G(\pm d,t)=0$ to be far from the potential minima, instead of natural boundary conditions (4). Note that we located reflecting boundaries far enough from the potential minima and controlled it carefully, thus even for the large noise intensity (indicated below) the influence of reflecting boundaries on the diffusion process was negligible. Comparing the computer simulation results with formula (16), we have substituted exact mean decay time $\tau(\varphi_0,\pi)$, Eq. (14), for asymptotic formula (15) and have found a really close coincidence between the curves, even for a noise intensity larger than unity, where formula (15) is not valid (see Fig. 1). Figure 1 presents the numerically simulated nondecay probability $P(\varphi_0,t)$ and the approximate one versus dimensionless time $t^*=\omega_c t$. The potential barrier height and the dimensionless inductance are, respectively, $\Delta u \approx 1.3$, $\ell=6$. The maximal difference δ between the corresponding curves is $\delta<0.4\%$, $\gamma=0.2$; $\delta<1\%$, $\gamma=0.3$; $\delta\approx 1\%$, $\gamma=0.4$; $\delta<1.5\%$, $\gamma=0.5$; $\delta<3\%$, $\gamma=1$; $\delta\approx 2\%$, $\gamma=2$; $\delta\approx 5\%$, $\gamma=5$; $\delta\approx 7\%$, $\gamma=10$.

V. CONCLUSIONS

In the present paper a complete description of noise-induced transitions in a bistable memory cell based on the

parametric quantron has been carried out. Exact values of the moments of decay time and an almost exact nondecay probability have been obtained. It has been demonstrated that the approximate model of exponential decay of the ''correct'' state of a memory cell is applicable with a good precision even for a large noise intensity (large enough for real applications, when the considered system cannot already be used for storage of information), if the approximate decay time is replaced by the exact one. The presented theory may be easily used for design and analysis of real devices.

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